MORPHISMS AND INVERSE PROBLEMS

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Abstract. In order to investigate polynomial vector fields admitting a prescribed Darboux integrating factor, we show that it is helpful to employ morphisms of the affine plane. In particular, such morphisms may be used to remove degeneracies of the underlying curve. Our main result states that the space of vector fields admitting a prescribed Darboux integrating factor modulo a well understood subspace has finite dimension. This extends earlier work for the case of nondegenerate geometric setting. In addition, we present a number of explicit examples with degenerate underlying curve.

1. Introduction and statement of the main result

This paper continues our work on inverse problems in the Darboux theory of integration. In a nondegenerate geometric setting, these inverse problems are well understood, see [2], [3]. Moreover, we showed in [2] that the general inverse problem for invariant algebraic curves can be reduced to standard tasks of algorithmic algebra. On the other hand, the algorithmic approach does not give much structural insight for the curve scenario, and for this reason sigma processes were also employed in [2]. In the present paper we will discuss the role of morphisms in solving and understanding inverse problems for Darboux integrating factors. As in previous articles, a characteristic feature will be that we work in the affine plane, thus degeneracies at infinity do not matter.

We consider a complex polynomial vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{1}$$

sometimes also written as $X = (P, Q)^t$, and a non-constant polynomial $f \in \mathbb{C}[x, y]$ with irreducible factors $f_1, \ldots, f_r$. The degree of f will be denoted by $\delta(f)$.

We know (cf. [2], [9]) that the complex zero set of f is invariant for the vector field if and only if there is a polynomial L (called the cofactor of f) such that

$$Xf = L \cdot f, \quad \text{or } P \cdot f_x + Q \cdot f_y = L \cdot f. \tag{2}$$

We will briefly say that in this case *the vector field X admits f*, or *admits the algebraic curve given by $f = 0$*. In the following, we will assume $f = f_1 \cdots f_r$, with no loss of generality. The respective zero sets of f and f_i in $\mathbb{C}^2$ will be denoted by C and C_i.

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As usual, we call a point z with $f(z) = f_x(z) = f_y(z) = 0$ a singular point of C, and similarly for the C_i. The *Hamiltonian vector field* of f is defined by

$$X_f = -f_y \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial y}.$$

We now fix $f_1, \ldots, f_r$. Then the vector fields admitting f form a linear space $\mathcal{V}$. To determine this space is equivalent to solving an inverse problem posed by Poincaré, i.e. to determine all the invariant algebraic curves for a given polynomial vector field. This inverse problem is quite well-understood and algorithmically accessible; see [9], [4], [5], [1], [2], [3]. The subspace of $\mathcal{V}$ which consists of all vector fields of the type

$$X = a \cdot X_f + f \cdot \widetilde{X} \tag{3}$$

(with a polynomial a and a polynomial vector field $\widetilde{X}$) will be called $\mathcal{V}^0$. Moreover the subspace of $\mathcal{V}$ which consists of all vector fields of the type

$$X = \sum a_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \widetilde{X} \tag{4}$$

(with polynomials a_i and a polynomial vector field $\widetilde{X}$) will be called $\mathcal{V}^1$. A central result of [9] and [2] is that $\mathcal{V}^0$ has finite codimension in $\mathcal{V}$. Following [5], two generic nondegeneracy conditions were introduced in [3]:

(ND1) Each C_i is nonsingular.

(ND2) All singular points of C have multiplicity one (thus when two irreducible components intersect, they intersect transversely, and no more than two irreducible components intersect at one point).

It was shown in [1] and in [2] that $\mathcal{V} = \mathcal{V}^1$ if (ND1) and (ND2) are satisfied.

In the present paper our principal focus is on the inverse problem for Darboux integrating factors. For the following fix complex constants $d_1, \ldots, d_r$, all of them nonzero. The vector fields with integrating factor

$$\left(f_1^{d_1} \cdots f_r^{d_r}\right)^{-1} \tag{5}$$

form a linear space $\mathcal{F}$, which is a subspace of $\mathcal{V}$. We first exhibit some of its elements; cf. [9] and [1]. Given an arbitrary polynomial g, define

$$Z_g = Z_g^{(d_1,\ldots,d_r)} \tag{6}$$

to be the Hamiltonian vector field of $g/\left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right)$. Then

$$f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g^{(d_1,\ldots,d_r)} = f X_g - \sum_{i=1}^{r} (d_i - 1) g \frac{f}{f_i} X_{f_i} \in \mathcal{F} \tag{7}$$

is easily verified. Note that the last expression is a polynomial vector field, and that the property of admitting the integrating factor (5) does not depend on the irreducibility or the relative primeness of the f_i. The vector fields of this particular type form a subspace $\mathcal{F}^0$ of $\mathcal{F}$. In presence of the geometric nondegeneracy conditions (ND1) and (ND2) we showed in [3] that the codimension of $\mathcal{F}^0$ in $\mathcal{F}$ is finite. The main result of the present paper is that this codimension is finite for any underlying geometry.

Theorem 1. *Let $f_1, \ldots, f_r$ be irreducible and pairwise relatively prime polynomials, and $d_1, \ldots, d_r$ nonzero constants. Then the dimension of $\mathcal{F}/\mathcal{F}^0$ is finite.*

The article is organized as follows. In Section 2 we establish some basic properties and technical results. In Section 3 we discuss properties of sigma processes and use them, in conjunction with the result from [3] for nondegenerate underlying geometry, to prove Theorem 1. In Section 4 we introduce morphisms related to planar reflection groups, and present a number of explicit examples with degenerate irreducible underlying curve.

2. Morphisms

Morphisms of the affine plane may transform certain degenerate geometric settings into nondegenerate ones, in the sense that (ND1) and (ND2) hold after transformation, or at least into less degenerate settings. Since the inverse problems for curves, resp. for integrating factors, are well understood in the nondegenerate setting, this provides a path to a better understanding in general. Such a strategy was employed in Section 5 of [2] to explicitly determine all polynomial vector fields that admit certain degenerate invariant curves.

Consider a polynomial map

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \det D\Phi \neq 0, \tag{8}$$

thus the image of Φ is dense in the plane (see e.g. Shafarevich [7]) and we have local analytic invertibility on an open and dense set. The comorphism of Φ assigns to every polynomial $g \in \mathbb{C}[x, y]$ the polynomial

$$\widehat{g} := g \circ \Phi, \tag{9}$$

and to every polynomial vector field $X = P\,\partial/\partial x + Q\,\partial/\partial y$ the rational vector field

$$\Phi_*(X) = D\Phi(x, y)^{-1} \begin{pmatrix} P\,(\Phi(x, y)) \\ Q\,(\Phi(x, y)) \end{pmatrix} \tag{10}$$

as well as the polynomial vector field

$$\widehat{X} = \det(D\Phi(x, y)) \cdot \Phi_*(X). \tag{11}$$

Note that these definitions also make sense for analytic functions and vector fields.

Proposition 2. *Let $g = g_1 \cdots g_r$ be a polynomial, with irreducible factors g_i, and X a polynomial vector field on $\mathbb{C}^2$.*

(a) *The zero set of g is invariant for X if and only if the zero set of $\hat{g}$ is invariant for $\widehat{X}$:*

$$X(g) = K \cdot g \Leftrightarrow \widehat{X}(\hat{g}) = \widehat{K} \cdot \hat{g} \quad \text{with } \widehat{K} := \det D\Phi \cdot (K \circ \Phi).$$

(b) *Given constants $d_1, \ldots, d_r$, the vector field X admits the integrating factor $g_1^{-d_1} \cdots g_r^{-d_r}$ if and only if the vector field $\widehat{X}$ admits the integrating factor $\hat{g}_1^{-d_1} \cdots \hat{g}_r^{-d_r} = (g_1^{-d_1} \cdots g_r^{-d_r}) \circ \Phi$.*

Proof. Part (a) follows directly, since by design

$$\Phi_*(X)\,(g \circ \Phi) = (X(g)) \circ \Phi.$$

For part (b) cf. [9], Corollary 1.3, to see that the factor $\det(D\Phi)$ enters the picture in case of a transformation. □

Some elementary properties of the transformations will be collected next, for easy reference.

Lemma 3. *Let Φ be as in (8).*

(a) *For any analytic f one has the identity*

$$\widehat{X_f} = X_{\widehat{f}}.$$

(b) *Given polynomials $f_1, \ldots, f_r$ and g, and the vector field defined in (6), one has the identity*

$$\hat{f}_1^{d_1} \cdots \hat{f}_r^{d_r} Z_{\widehat{g}} = f_1^{d_1} \widehat{\cdots f_r^{d_r}} Z_g.$$

Proof. We supply a direct proof of (a) for the sake of completeness; part (b) is then immediate from the definitions. We have

$$X_f(x,y) = J \cdot Df(x,y)^t, \quad \text{with } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $.^t$ denotes transposition. Using the identity

$$J \cdot A^t = A^* \cdot J$$

for arbitrary matrices A, with A^* the adjoint of A, and

$$D\hat{f}(x,y)^t = D\Phi(x,y)^t \cdot Df(x,y)^t,$$

we find

$$X_{\hat{f}}(x,y) = \det D\Phi(x,y)\, D\Phi(x,y)^{-1} \cdot J \cdot Df(\Phi(x,y))^t$$

as asserted. □

Remark 4. *The following statement hold.*

(a) *In particular Proposition 2 and Lemma 3 apply to automorphisms of the affine plane. Mutatis mutandis, results about vector fields that admit invariant curves or Darboux integrating factors are unaffected by automorphisms. This fact has been tacitly used in* [3], *for instance. But note that automorphisms will not remove degeneracies.*

(b) *To apply Proposition 2 and Lemma 3, we would like to start with a morphism Φ that turns a polynomial f, with degenerate geometry of the underlying curve C, to a polynomial $\widehat{f}$ with nondegenerate, or less degenerate, geometry of the underlying curve. For the transformed polynomial we may be able to determine all vector fields that admit $\widehat{f}$, resp. a particular Darboux integrating factor. There remains the problem to decide under what circumstances such a vector field Y is of the type $\widehat{X}$ as given in (11). Generally this problem is nontrivial, but in the following sections we consider classes of morphisms for which it is manageable.*

(c) *In Lemma 3(b), some $\hat{f}_i$ may be reducible even if the f_i are irreducible. Note that the appearance of $\hat{f}_1^{d_1} \cdots \hat{f}_r^{d_r} Z_{\widehat{g}}$ will change when it is rewritten in the form (7) with irreducible factors.*

3. Sigma processes

The first class of morphisms we will discuss are sigma processes. The discussion will not aim at investigating explicit examples but rather at obtaining a general finiteness result.

In the affine plane the basic sigma process with center 0 and direction given by $x = 0$ is represented by the birational morphism

$$\Phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ xy \end{pmatrix}.$$

(Generally, a sigma process will be composed of a basic sigma process, a linear transformation and a translation.) This morphism induces a map from polynomials to polynomials, sending g to $\hat{g}$ with $\hat{g}(x,y) = g(x,xy)$; and a map ι from polynomial vector fields to polynomial vector fields, sending X to $\widehat{X}$, with

$$\widehat{X} = \begin{pmatrix} xP(x,xy) \\ -yP(x,xy) + Q(x,xy) \end{pmatrix} \quad \text{if} \quad X = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}.$$

Both maps are linear and injective. We collect some useful criteria; these were also employed in [2].

Lemma 5. *The following statement hold.*

(a) *Let*

$$h = \sum x^\ell h_\ell(y)$$

be a polynomial. Then $h = \hat{g}$ for some g if and only if $\delta(h_\ell) \le \ell$ for all ℓ.

(b) *Let $Y = (R, S)$ be a vector field, and*

$$R = \sum x^\ell v_\ell^*(y), \quad yR + xS = \sum x^\ell w_\ell^*(y).$$

Then $Y = \widehat{X}$ for some X if and only if $\delta(v_\ell^*) \le \ell - 1$ and $\delta(w_\ell^*) \le \ell - 1$ for all ℓ.

According to Proposition 2, if X admits the integrating factor $f_1^{-d_1} \cdots f_r^{-d_r}$ then $\widehat{X}$ admits the integrating factor $\hat{f}_1^{-d_1} \cdots \hat{f}_r^{-d_r}$. It is appropriate to rewrite this with irreducible factors. Letting

$$\hat{f}_i = x^{s_i} \cdot f_i^*$$

with $s_i \ge 0$ and irreducible f_i^* we obtain

$$\hat{f}_1^{-d_1} \cdots \hat{f}_r^{-d_r} = x^{-(d_1 s_1 + \ldots + d_r s_r)} f_1^{*-d_1} \cdots f_r^{*-d_r}. \tag{12}$$

We will denote the space of vector fields with the integrating factor (12) by $\widehat{\mathcal{F}}$, with corresponding subspace $\widehat{\mathcal{F}}^0$. Lemma 3 implies

$$\iota(\mathcal{F}^0) \subseteq \widehat{\mathcal{F}}^0.$$

A priori there may be vector fields in $\iota(\mathcal{F}) \cap \widehat{\mathcal{F}}^0$ which are not in $\iota(\mathcal{F}^0)$. The following auxiliary result clarifies these matters to some extent.

Lemma 6. *The vector field*

$$\begin{aligned} Y \quad = \quad & x \cdot f_1^* \cdots f_r^* X_h - \sum_{i=1}^r (d_i - 1) xh \cdot \frac{f_1^* \cdots f_r^*}{f_i^*} X_{f_i^*} \\ & -(\sum_{i=1}^r s_i d_i - 1) h \cdot f_1^* \cdots f_r^* X_x \quad \in \quad \widehat{\mathcal{F}}^0 \end{aligned}$$

lies in $\iota(\mathcal{F}^0)$ if

$$h = x^{s_1 + \ldots + s_r - 1} \cdot \hat{g}$$

for some g. This condition is equivalent to

$$h = \sum x^\ell h_\ell(y), \quad \text{with } \delta(h_\ell) \le \ell - s_1 - \ldots - s_r + 1.$$

Proof. If $h = x^{s_1 + \ldots s_r - 1} \cdot \hat{g}$ then

$$\begin{aligned} Y &= (s_1 + \cdots + s_r - 1) x^{s_1 + \cdots + s_r - 1} f_1^* \cdots f_r^* \hat{g} X_x \\ &\quad + x^{s_1 + \cdots + s_r} f_1^* \cdots f_r^* X_{\hat{g}} \\ &\quad - \sum_{i=1}^r (d_i - 1) x^{s_1 + \cdots + s_r - 1} \hat{g} \cdot \frac{f_1^* \cdots f_r^*}{f_i^*} X_{f_i^*} \\ &\quad - (\sum_{i=1}^r s_i d_i - 1) x^{s_1 + \cdots + s_r} \hat{g} \cdot f_1^* \cdots f_r^* X_x \\ &= x^{s_1 + \cdots + s_r} f_1^* \cdots f_r^* X_{\hat{g}} \\ &\quad - \sum_{i=1}^r (d_i - 1) x^{s_1 + \cdots + s_r} \hat{g} \cdot \frac{f_1^* \cdots f_r^*}{f_i^*} X_{f_i^*} \\ &\quad - (\sum_{i=1}^r s_i (d_i - 1)) x^{s_1 + \cdots + s_r} \hat{g} \cdot f_1^* \cdots f_r^* X_x . \end{aligned}$$

Using (7), one verifies directly that this is equal to

$$\hat{f}_1^{d_1} \cdots \hat{f}_r^{d_r} \cdot Z_{\hat{g}}.$$

□

We recall that our aim is to show that $\mathcal{F}/\mathcal{F}^0$ is finite dimensional if $\widehat{\mathcal{F}}/\widehat{\mathcal{F}}^0$ is. The first step is as follows.

Lemma 7. *If*

$$\dim \widehat{\mathcal{F}}/\widehat{\mathcal{F}}^0 < \infty \text{ and } \dim \left(\iota(\mathcal{F}) \cap \widehat{\mathcal{F}}^0 \right) / \iota(\mathcal{F}^0) < \infty,$$

then

$$\dim \mathcal{F}/\mathcal{F}^0 < \infty.$$

Proof. We assume that

$$\dim \widehat{\mathcal{F}}/\widehat{\mathcal{F}}^0 < \infty \text{ and } \dim \mathcal{F}/\mathcal{F}^0 = \infty,$$

and let $X_i \in \mathcal{F}$, $i = 1, 2, \ldots$ be an infinite system such that $(X_i + \mathcal{F}^0)_{i \ge 1}$ is linearly independent. Since $\dim \widehat{\mathcal{F}}/\widehat{\mathcal{F}}^0 < \infty$, we may assume after possible relabeling that there is some $q \ge 1$ such that $\left(\widehat{X}_1 + \widehat{\mathcal{F}}^0, \ldots, \widehat{X}_{q-1} + \widehat{\mathcal{F}}^0 \right)$ is a linearly independent system but $\widehat{X}_1 + \widehat{\mathcal{F}}^0, \ldots, \widehat{X}_{q-1} + \widehat{\mathcal{F}}^0, \widehat{X}_{q-1+i} + \widehat{\mathcal{F}}^0$ are linearly dependent for every $i \ge 1$. Therefore there exist scalars β_{ij} such that

$$\widehat{X}_{q-1+i} - \sum_{j=1}^{q-1} \beta_{ij} \widehat{X}_j \in \widehat{\mathcal{F}}^0, \quad \text{all } i \ge 1.$$

We define

$$Y_i := X_{q-1+i} - \sum_{j=1}^{q-1} \beta_{ij} X_j \in \mathcal{F}, \; i \ge 1.$$

Then $(Y_i + \mathcal{F}^0)$ is an infinite linearly independent system. Since ι is injective, the system $(\widehat{Y}_i + \iota(\mathcal{F}^0))$ is also linearly independent. Therefore

$$\dim \left(\iota(\mathcal{F}) \cap \widehat{\mathcal{F}}^0 \right) / \iota(\mathcal{F}^0) = \infty.$$

$\square$

Now we write

$$f_i^* = \sum x^\ell f_{i,\ell}(y), \tag{13}$$

and note that $\delta(f_{i,\ell}) \leq s_i + \ell$ by construction. Moreover we may assume that $\delta(f_{i,0}) = s_i$: This can be achieved via a linear automorphism $x \mapsto x + \alpha y$, $y \mapsto y$ with suitable α. In other words, this can be achieved by suitable choice of direction of the sigma process, and only finitely many directions have to be excluded. Moreover, we are only interested in the case of a degenerate singular point at 0, thus $\sum s_i \geq 2$.

Lemma 8. *If $\delta(f_{0,i}) = s_i$ for $i = 1, \ldots, r$, and $\sum s_i \geq 2$, then*

$$\iota(\mathcal{F}) \cap \widehat{\mathcal{F}}^0 = \iota(\mathcal{F}^0).$$

Proof. (i) We consider the vector field $Y \in \widehat{\mathcal{F}}^0$ as in Lemma 6, and we abbreviate $Y = (R, S)$. Then

$$R = x\left(-f_1^* \cdots f_r^* h_y + (\sum_{i=1}^r (d_i - 1)\frac{f_1^* \cdots f_r^*}{f_i^*} f_{i\,y}^*)h\right) = x \cdot \sum x^\ell v_\ell(y). \tag{14}$$

According to Lemma 5 we have $Y \in \iota(\mathcal{F})$ only if $\delta(v_\ell) \leq \ell$ for all ℓ. Moreover we have

$$\begin{array}{rcl} yR + xS & = & yR \\ & + & x\left(f_1^* \cdots f_r^* x h_x - \sum (d_i - 1) h \dfrac{f_1^* \cdots f_r^*}{f_i^*} x f_{i\,x}\right) \\ & & -x\left((\sum s_i d_i - 1) h f_1^* \cdots f_r^*\right) \end{array}$$

According to Lemma 5 we have $Y \in \iota(\mathcal{F})$ only if $yR + xS$ is of the form

$$x \cdot \sum x^\ell \tilde{w}_\ell(y), \quad \delta(\tilde{w}_\ell) \leq \ell.$$

Combining this with the condition on R, we find that $Y \in \iota(\mathcal{F})$ only if

$$\begin{array}{rcl} f_1^* \cdots f_r^* x h_x - \sum (d_i - 1) h \dfrac{f_1^* \cdots f_r^*}{f_i^*} x f_{i\,x} & & \\ -(\sum s_i d_i - 1) h f_1^* \cdots f_r^* & = & \sum x^\ell w_\ell \end{array} \tag{15}$$

with $\delta(w_\ell) \leq \ell + 1$ for all ℓ. We will evaluate (14) and (15) degree by degree in x. Thus we write

$$h = \sum x^\ell h_\ell(y)$$

and note that

$$h_y = \sum x^\ell h_\ell'(y), \quad x h_x = \sum \ell x^\ell h_\ell(y).$$

From (13) we have

$$f_{i\,y}^* = \sum x^\ell f_{i,\ell}'(y), \quad x f_{i\,x}^* = \sum \ell x^\ell f_{i,\ell}(y).$$

The degree by degree evaluation of (14) yields

$$
\begin{aligned}
v_\ell &= -f_{1,0}\cdots f_{r,0}\cdot h_\ell' + (\sum_{i=1}^{r}(d_i-1)\frac{f_{1,0}\cdots f_{r,0}}{f_{i,0}}f_{i,0}{}')\cdot h_\ell \\
&\quad - \sum_{j_1,\dots,j_r} f_{1,j_1}\cdots f_{r,j_r}h'_{\ell-(j_1+\cdots j_r)} \\
&\quad + \sum_{j_1,\dots,j_r}\sum_i (d_i-1)\frac{f_{1,j_1}\cdots f_{r,j_r}}{f_{i,j_i}}f'_{i,j_i}h_{\ell-(j_1+\cdots j_r)},
\end{aligned} \tag{16}
$$

where the summation extends over all tuples $(j_1,\dots,j_r)$ of nonnegative integers such that $1\le\sum j_i\le\ell$. The degree by degree evaluation of (15) yields

$$
\begin{aligned}
w_\ell &= (\ell-\textstyle\sum_i s_id_i+1)f_{1,0}\cdots f_{r,0}\cdot h_\ell \\
&\quad + \sum_{j_1,\dots,j_r}(\ell-\sum_i d_ij_i-\sum_i d_is_i+1)f_{1,j_1}\cdots f_{r,j_r}h_{\ell-(j_1+\cdots j_r)},
\end{aligned} \tag{17}
$$

with the same range for $(j_1,\dots,j_r)$.

(ii) Given a nonzero polynomial p, the polynomial

$$
-f_{1,0}\cdots f_{r,0}\cdot p' + (\sum_{i=1}^{r}(d_i-1)\frac{f_{1,0}\cdots f_{r,0}}{f_{i,0}}f_{i,0}{}')\cdot p
$$

has degree $<\delta(p)+s_1+\cdots+s_r-1$ only if

$$
\delta(p)=\sum s_i(d_i-1).
$$

To see this we note that the coefficient of the term of degree $\delta(p)+s_1+\cdots+s_r-1$ is equal to the product of the leading coefficients of the $f_{i,0}$ and p, and the factor

$$
-\delta(p)+\sum s_i(d_i-1).
$$

(iii) We will prove by induction on ℓ: If the vector field Y from Lemma 6 lies in $\iota(\mathcal{F})$ then

$$
\delta(h_\ell)\le\ell-\sum s_i+1,
$$

with the tacit understanding that $\delta(h_\ell)<0$ means $h_\ell=0$. We will use the degree conditions in (16) and (17).

$\ell=0$: In the case that $\sum s_id_i-1\ne0$, the assumption $h_0\ne0$ and (17) lead to

$$
s_1+\cdots+s_r+\delta(h_0)=\delta(f_{1,0}\cdots f_{r,0}\cdot h_0)\le1;
$$

a contradiction. In the case $\sum s_id_i-1=0$, the assumption $h_0\ne0$, part (ii) and the degree condition in (16) lead to

$$
\delta(h_0)=\sum s_id_i-\sum s_i=1-\sum s_i<0;
$$

which also gives a contradiction.

For the induction step we assume that the assertion holds for all $h_{\ell-j}$, with $1\le j\le\ell$. Since the degree of f_{i,j_i} is at most equal to s_i+j_i, every term on the right-hand side of (16), with the possible exception of those involving h_ℓ, has degree $\le\ell$. By the same argument, every term on the right-hand side of (17), with the possible exception of those involving h_ℓ, has degree $\le\ell+1$. Therefore we have

$$
\delta\left(-f_{1,0}\cdots f_{r,0}\cdot h_\ell' + (\sum_{i=1}^{r}(d_i-1)\frac{f_{1,0}\cdots f_{r,0}}{f_{i,0}}f_{i,0}{}')\cdot h_\ell\right)\le\ell
$$

as well as

$$\delta\left((\ell - \sum_i s_i d_i + 1) f_{1,0} \cdots f_{r,0} \cdot h_\ell\right) \leq \ell + 1$$

by the induction hypothesis. In the case that $\ell \neq \sum s_i d_i - 1$, the second condition directly shows $\delta(h_\ell) \leq \ell - \sum s_i + 1$, as desired. In the case $\ell = \sum s_i d_i - 1$, the assumption $\delta(h_\ell) > \ell - \sum s_i + 1$ implies that highest-degree terms in (16) must cancel. By part (ii) this implies

$$\delta(h_\ell) = \sum s_i d_i - \sum s_i = \ell - \sum s_i + 1,$$

an obvious contradiction. □

This leads us to prove the main result of the present paper.

Proof of Theorem 1. The Bendixson-Seidenberg theorem, cf. Seidenberg [6], for X_f and Lemma 3 show that a finite number of sigma processes, with suitable centers and directions, will transform the f_i to polynomials $\tilde{f}_i$ which satisfy the nondegeneracy conditions (ND1) and (ND2). In every single sigma process at most finitely many directions have to be excluded, as can be seen from [6]. For $\tilde{f}_1, \ldots, \tilde{f}_r$ and $d_1, \ldots, d_r$ finiteness holds by Theorem 11 and Theorem 3 of [3].

For a single sigma process at a degenerate singular point, with a suitably chosen direction, Lemma 7 and Lemma 8 show that finiteness holds for the original setting if it holds for the transformed polynomials. Induction on the number of sigma processes finishes the proof. □

Remark 9. . *Classically, one knows that sigma processes can be used directly to simplify singular points of curves; see Shafarevich* [7]*, Ch. II, §4 (including Exercises). We chose the detour via vector fields because Seidenberg gives a complete proof which shows clearly that the exclusion of finitely many directions does not matter.*

Theorem 1 is quite satisfactory from a theoretical perspective, but it is not directly applicable to explicit computations in examples. While the information in Lemmas 7 and 8 is directly useful for computations, one only has Lemma 5 available for computations to determine vector fields not in $\mathcal{F}^0$. As Section 5 in [2] illustrates, this may amount to nontrivial work. For this reason other classes of morphisms are of interest.

4. Invariants of reflection groups

In this section we discuss morphisms related to reflection groups, which allow the investigation of some complicated geometric settings with little computational effort. In particular, the problem to decide whether $Y = \widehat{X}$ for some X, see Remark 4(b) is accessible. Recall that a reflection (in the sense of Chevalley) of the plane is a linear transformation with one eigenvalue 1 and one eigenvalue $\neq 1$. For more information we refer to Sturmfels [8]. We are interested in finite groups generated by reflections, hence the second eigenvalue is necessarily a nontrivial root ζ of unity. Up to an invertible linear transformation, such a reflection is therefore given by

$$T = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

Lemma 10. *Let T be a reflection, and let the nontrivial eigenvalue ζ be a primitive m^{th} root of unity. Then:*

(a) *T acts on polynomial functions via $f \mapsto f \circ T$. The space of polynomial functions is a direct sum of eigenspaces for this action, with eigenvalues $1, \zeta, \ldots, \zeta^{m-1}$.*
(b) *T acts on polynomial vector fields via $X \mapsto T^{-1}X \circ T$. The space of polynomial vector fields is a direct sum of eigenspaces for this action, with eigenvalues $1, \zeta, \ldots, \zeta^{m-1}$.*
(c) *If f is a polynomial such that $f \circ T = \zeta^r f$ then $T^{-1}X_f \circ T = \zeta^{r-1}X_f$.*

Proof. Parts (a) and (b) are immediate when T is given in the form (18), which is sufficient. Part (c) follows from Lemma 3 (a) with $\Phi = T$. □

Let G be a finite reflection group in the plane. A characteristic property of reflection groups is that their invariant algebra admits an algebraically independent set of generators (Chevalley's theorem; see Sturmfels [8]). In the planar case this means that there are two algebraically independent polynomials which generate the invariant algebra of G. Now consider a morphism Φ which has as components such a generating set for the invariant algebra of G. (For instance, if G is generated by T as given in (18) then $\Phi(x, y) = (x^m,\, y)$.)

In the following we abbreviate $z := (x,\, y)^t$. Since Φ is built from invariants of G, we have

$$\Phi(Tz) = Tz, \quad D\Phi(Tz)T = D\Phi(z), \quad \text{forall } T \in G. \tag{19}$$

Lemma 11. *Let G and Φ be as above. Then the following hold:*
(a) *Given a polynomial h, there exists a polynomial f such that $h = \widehat{f}$ if and only if $h \circ T = h$ for all $T \in G$.*
(b) *Given a vector field Y, there exists a vector field X such that $Y = \widehat{X}$ if and only if*
$$T^{-1}Y \circ T = (\det T)^{-1} \cdot Y,$$
for all $T \in G$.
(c) *If f is an irreducible polynomial such that $g := \widehat{f}$ is reducible, thus $g = g_1 \cdots g_s$ with irreducible g_i, then s divides the order m of G and the G-orbit of g_1 equals $\{g_1, \ldots, g_s\}$, up to multiplication by nonzero constants.*

Proof. (a) If $h = f \circ \Phi$ then for all z one has
$$h(Tz) = f(\Phi(Tz)) = f(\Phi(z)) = h(z),$$
whence $h \circ T = h$, for all $T \in G$. Conversely, $h \circ T = h$ for all T means that h is invariant for G, hence by construction of Φ there is an f such that $h = f \circ \Phi$. One direction of part (b) is obvious from (11) for the morphism T. For the converse direction assume that the identity holds and set
$$V(z) := (\det D\Phi(z))^{-1}\ D\Phi(z)Y(z).$$
A direct computation shows
$$T^{-1}V \circ T(z) = T^{-1}V(z),$$
for all $T \in G$ and all z. Therefore the entries of V are G-invariant and there is a vector field X such that $V = X \circ \Phi$. As for part (c), note that composition with T leaves g unchanged, and thus permutes the irreducible factors (up to nonzero constants). The product q of the elements in the orbit of, say, g_1 satisfies $q \circ T = q$, and therefore $q = p \circ \Phi$ for some p. Moreover, p divides f because q divides g. Since f is irreducible,

p must be a constant multiple of f, whence the orbit of g_1 contains all irreducible factors. □

Remark 12. *One may turn Lemma 11(c) around to construct examples. Given a reflection group G, let p be an irreducible polynomial such that its G-orbit contains $|G|$ pairwise relatively prime polynomials $p = g_1, \ldots, g_m$. Then $g = \prod g_i$ is G-invariant by construction, hence $g = f \circ \phi$ for some polynomial f. Since the orbit of p has length $|G|$, f must be irreducible. If g satisfies the nondegeneracy conditions (ND1) and (ND2), Proposition 13 below is applicable for f.*

The main result of this section shows that degeneracies for the underlying curves may preclude the existence of nontrivial vector fields with a given integrating factor.

Proposition 13. *Let G and Φ be as above, and let f be an irreducible polynomial such that $\widehat{f} = g = g_1 \cdots g_s$ is reducible. Moreover, assume that g satisfies the nondegeneracy conditions (ND1) and (ND2). Then:*

(a) *The vector field X admits the integrating factor f^{-1} if and only if*
$$X = \alpha \cdot X_f + f \cdot \widetilde{X} \quad (\alpha \in \mathbb{C},\ \operatorname{div} \widetilde{X} = 0).$$

(b) *Given an integer $d > 1$, the vector field X admits the integrating factor f^{-d} if and only if*
$$X = f^d \left(\frac{\alpha}{f} \cdot X_f + Z_q^{(d)} \right)$$
for some $\alpha \in \mathbb{C}$ and some polynomial q.

Proof. By [3], Theorem 3, a vector field Y admits the integrating factor g^{-1} if and only if
$$Y = \sum_i \alpha_i \frac{g}{g_i} \cdot X_{g_i} + g \cdot X_h$$
with complex constants α_i and some polynomial h. The α_i and h are uniquely determined in this representation. (To verify uniqueness, consider the case $Y = 0$ and check prime factors.) Now one verifies
$$T^{-1} Y \circ T = \sum_i \alpha_{\pi_T(i)} \frac{g}{g_i} \cdot X_{g_i} + g \cdot X_{h \circ T}$$
for all $T \in G$, where π_T is a permutation of the indices which is defined by $g_{\pi_T}(i) \circ T = g_i$.

Evaluating the condition $T^{-1} Y \circ T = (\det T)^{-1} \cdot Y$ from Lemma 11, using uniqueness and Lemma 3, shows that $h \circ T = h$ as well as $\alpha_{\pi_T(i)} = \alpha_i$ for all i, for every $T \in G$. Since the G-orbit of g_1 contains all prime factors of g, one finds $\alpha_1 = \cdots = \alpha_s$. The assertion of part (a) now follows from Lemma 3. Part (b) is a direct consequence of (a) and the reduction principle given in [3], Lemma 2 and Lemma 6. □

Remark 14. *The following statements hold.*

(a) *Given a constant d which is not a positive integer, vector fields admitting the integrating factor g^{-d} may not always be of the form (6), although* [3] *indicates that exceptions are rare. But if a vector field Y admitting g^{-d} is of the form*
$$Y = g^d \cdot Z_h^{(d)}$$
for some polynomial h, then there exists a vector field X such that $Y = \widehat{X}$ if and only if
$$X = f^d \cdot Z_p^{(d)}$$

for some polynomial p. To verify this, write $h = \sum h_\ell$ with $h_\ell \circ T = \zeta^\ell h_\ell$ and note that

$$T^{-1} Z_{h_\ell} \circ T = \zeta^{\ell-1} Z_{h_\ell}$$

by Lemma 3(a). Therefore one may conclude $\mathcal{F} = \mathcal{F}^0$ if the corresponding property holds for g.

(b) *One can extend the argument in the proof to construct vector fields admitting f: Let*

$$W = \sum_i a_i \frac{g}{g_i} \cdot X_{g_i}$$

with the property that $a_{\pi_T}(i) \circ T = a_i$ for all i. Then $T^{-1} W \circ T = (\det T)^{-1} \cdot W$ holds for all $T \in G$, and therefore $W = \widehat{Z}$ for some Z. If one chooses a_1 that is not G-invariant then one will obtain vector fields admitting f which are not contained in $\mathcal{V}^0 (= \mathcal{V}^1)$. This direct approach to the construction of nontrivial vector fields admitting f is also a particular feature of morphisms related to reflection groups.

Example 1. Given a nonconstant polynomial q in one variable, with simple roots $v_1, \ldots, v_m$ that are all different from 0, consider

$$f = y^2 - x \cdot q(x)^2.$$

The polynomial f is irreducible, e.g. by Eisenstein's criterion. Now let

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}.$$

Then

$$g(x,y) := \hat{f}(x,y) = \Big(y - x \cdot q(x^2)\Big) \cdot \Big(y + x \cdot q(x^2)\Big) =: g_1 \cdot g_2$$

is reducible and the nondegeneracy conditions (ND1), (ND2) apply. This example fits into the general scheme from Remark 12 with the group G generated by the reflection T about the y-axis, and $p = g_1$.

We first discuss vector fields admitting f. The singular points are precisely the $z_i = (v_i, 0)$, $1 \le i \le m$, and from their Hessian we see that all these points have multiplicity 1. Hence the quotient space $\mathcal{V}/\mathcal{V}^0$ has dimension m, according to [2], Theorem 8. The vector field

$$W := -x g_2 \cdot X_{g_1} + x g_1 \cdot X_{g_2}$$

admits g and satisfies $T^{-1} W \circ T = -W$, hence is of the form $W = \widehat{Z}$ for some Z. (Its construction follows Remark 14(b) with $a_1 = -x$.) A straightforward computation shows

$$Z = 2x \cdot q(x) \partial/\partial x + y(2x \cdot q'(x) + q(x)) \partial/\partial y.$$

The cofactor of Z is equal to $4x \cdot q'(x) + 2q(x)$ and hence does not vanish at any singular point. Now the argument from [2], Theorem 8 and its proof shows that

$$\mathcal{V}_f = \mathcal{V}_f^0 + \{b \cdot Z;\, b \in \mathbb{C}[x,y]\}.$$

Let us turn to integrating factors. According to Proposition 13, for any positive integer d the vector field X admits the integrating factor f^{-d} if and only if

$$X = f^d \cdot \left(\frac{\alpha}{f} \cdot X_f + Z_q^{(d)} \right)$$

with some constant α and some polynomial q.

Finally, given a constant d which is not a positive integer, the vector field X admits the integrating factor f^{-d} if and only if

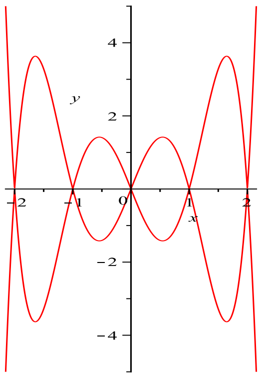
$$X = f^d \cdot Z_p^{(d)}$$

for some polynomial p. To see this we note that a vector field Y admits the integrating factor g^{-d} if and only if

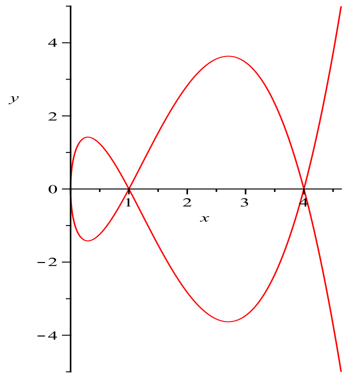
$$Y = g^d \cdot Z_h^{(d)}$$

for some polynomial h, due to Theorem 24 (b) of [3]. Remark 14 now shows that h is T-invariant.

For the purpose of illustration we consider a concrete example, with $q(x) = (1-x)(4-x)$.



$g = (y - x(1-x^2)(4-x^2))(y + x(1-x^2)(4-x^2))$.



$f = y^2 - x(1-x)^2(4-x)^2$.

FIGURE 1. The zero sets of the reducible polynomial g and the irreducible polynomial f.

Figure 1 shows the zero set of the reducible polynomial g on the left, and the zero set of the irreducible polynomial f on the right, which is just the image of the former with respect to Φ.

Example 2. Let q_1, q_2 be nonconstant polynomials in one variable with $q_1(0) \neq 0$, $q_2(0) \neq 0$, and

$$p = y + q_2(y^2) - xq_1(x^2).$$

Moreover let G be the four-element group generated by the reflections T_1 about the y-axis and T_2 about the x-axis. According to Lemma 11(c) and Remark 12, compute $g_1 = p$,

$$g_2 = y + q_2(y^2) + xq_1(x^2), \quad g_3 = -y + q_2(y^2) - xq_1(x^2), \quad g_4 = -y + q_2(y^2) + xq_1(x^2),$$

and define

$$g = g_1 \cdots g_4.$$

With

$$\Phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

one obtains the irreducible polynomial

$$f = (y - q_2(y)^2)^2 - xq_1(x^2)(y + q_2(y)^2) + x^2 q_1(x)^4,$$

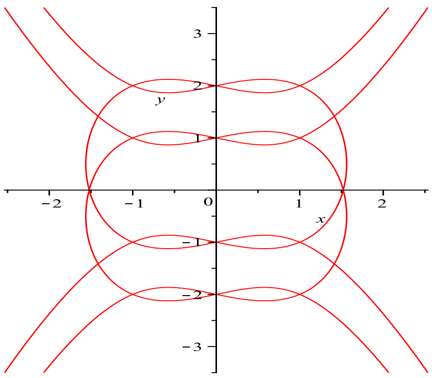
with $g = f\circ\Phi$. Now fix q_1 and assume that it has only simple roots, hence $x \mapsto x\cdot q_1(x^2)$ has only simple roots. Moreover write $q_2 = v + \beta$ with a constant $\beta \neq 0$ and $v(0) = 0$. Then there are only finitely many values of β such that the curve $\{p = 0\}$ is not smooth, because the gradient of p has only finitely many zeros and is independent of β. Similarly, there are only finitely many values of β such that $y \mapsto q_2(y^2)$ or $y \mapsto y+q_2(y^2)$ or $y \mapsto y - q_2(y^2)$ has a multiple root. Excluding these exceptional values, one verifies by straightforward computation: The only singular points on the curve $\{g = 0\}$ are intersection points of two curves $\{g_i = 0\}$. There are no triple intersections, and the intersections are transversal. To summarize, with the exception of finitely many values for β, Proposition 13 is applicable, and we have found in a simple manner, for quite complicated-looking geometry, all the vector fields admitting the integrating factor f^{-d}, for d a positive integer. The case when d is not a positive integer cannot be discussed generally, since there are no general results available for the reducible polynomial g.

As for vector fields admitting f, by Remark 14(b) we define

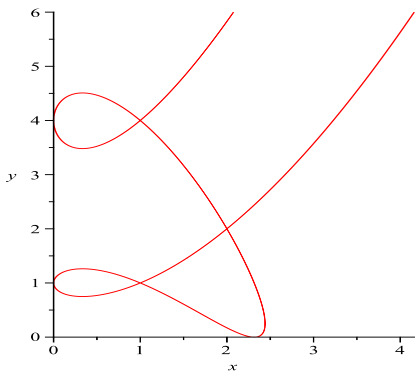
$$\begin{aligned} W_1 &:= \frac{xg}{g_1}\cdot X_{g_1} - \frac{xg}{g_2}\cdot X_{g_2} + \frac{xg}{g_3}\cdot X_{g_3} - \frac{xg}{g_4}\cdot X_{g_4}, \\ W_2 &:= \frac{yg}{g_1}\cdot X_{g_1} + \frac{yg}{g_2}\cdot X_{g_2} - \frac{yg}{g_3}\cdot X_{g_3} - \frac{yg}{g_4}\cdot X_{g_4}, \end{aligned}$$

with $a_1 = x$ resp. $a_1 = y$. Then $W_i = \widehat{Z}_i$ for suitable Z_i, and an elementary verification similar to the one in Example 1 shows that $\mathcal{V}$ is spanned by $\mathcal{V}^0$ and polynomial multiples of the Z_i. Thus, although the geometry is complicated, we obtain the vector fields admitting f with little effort.

Again we consider a concrete example for illustration, with $q_1(x) = 1-x$ and $q_2(y) = y - 2$.



The graph of $g_1g_2g_3g_4 = g = 0$.



The graph of f=0.

FIGURE 2. The zero set of the reducible polynomial g and the irreducible polynomial f. Note that $g(x, y) = \hat{f}(x, y)$.

Figure 2 shows the zero set of the reducible polynomial g on the left, and the zero set of the irreducible polynomial f on the right, which is just the image of the former with respect to Φ.

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